Some Random Stuff About Random Parking

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Generalizing Parking Functions with Randomness

Introduction

- Consider a parking lot with spots labeled 1 to n.
- n cars go into the parking lot 1 by 1 trying to park.
- Each car c_i has a preference spot a_i.
- Now we have a preference list $(a_1, a_2, ..., a_n)$.
- If c_i's preference spot a_i is taken (by c_j, $j < i$), then c_i goes forward searching for the next available spot to park.
- If a preference list $(a_1, a_2, ..., a_n)$ let every car park, it becomes a Parking function.

Examples

$n = 4$

Some stuff that work:

- $(1, 2, 3, 4)$
- $(1, 2, 2, 4)$
- $(4, 1, 1, 3)$

Some stuff that do not work:

- $(1, 3, 3, 4)$
- $(4, 4, 4, 4)$
- $(2, 2, 2, 2)$

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Surprise. *The number of Parking functions of length n is $(n + 1)^{n-1}$.*

Introducing Randomness

Our first generalization

- if a_i is taken, now c_i flips a fair coin.
- If it's head, c_i goes forward, and backwards if it's tail.
- Once c_i has chosen that direction, no turning back.

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While some of the stuff that work before has a even lower probability.

- $(1, 2, 2, 4)$, probability $\frac{1}{2}$.

$n = 3$

- (1,1,1), probability $\frac{1}{4}$.
- (1,1,2), probability $\frac{1}{4}$.
- (1,1,3), probability $\frac{1}{2}$.
- (1,2,1), probability $\frac{1}{2}$.
- (1,2,2), probability $\frac{1}{2}$.
- (1,2,3), probability 1.
- (1,3,1), probability $\frac{1}{2}$.
- (1,3,2), probability 1.
- (1,3,3), probability $\frac{1}{2}$.
- (2,1,1), probability $\frac{1}{2}$.
- (2,1,2), probability $\frac{1}{2}$.
- (2,1,3), probability 1.
- (2,2,1), probability $\frac{3}{4}$.
- (2,2,2), probability $\frac{1}{2}$.

Once passed (2,2,2), it's all symmetric.

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Real Big Surprise. *This is the exact same number as Parking functions.*

Our Second Generalization

- Whenever stuck, flip a coin (not necessarily fair) to choose a direction.
- Going backwards means backing up one spot at a time up to k spots to check for an available spot before they go forward to check for the next available spot.

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$n = 4, p = \frac{1}{2}, k = 1$ examples:

- (1,3,3,4), probability: $\frac{1}{2}$
- (2,2,2,2), probability: $\frac{7}{8}$

$p = \frac{1}{2}, k = 1$ Expected Value

n	1	2	3	4	5	6	7	8	9
EV	1	3.5	20	163.25	1744.25	23121.375	366699	6779029.0625	143242152.5625

Table: Expected values for $n < 10$.

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Table: Expected values for $n < 10$.

No one:

Me: I'm gonna guess a formula for this.

We Guessed A Formula

Let $T_{k,p}(n)$ be the number of Parking functions.

$$T_{k,p}(n) = \sum_{i=0}^{n-1} \binom{n-1}{i} T_{k,p}(i)(n-i)^{n-i-2}(i+1+p\min\{k, n-i-1\})$$

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No one:

Me: Beautiful.

Distribution of Probabilities, $n = 7$ Example

p	0	1/64	2/64	3/64	4/64	5/64	6/64	7/64
$f(p)$	339472	1	136	1	2194	1	209	1
p	8/64	9/64	10/64	11/64	12/64	13/64	14/64	15/64
$f(p)$	12466	1	140	1	3107	1	143	1
p	16/64	17/64	18/64	19/64	20/64	21/64	22/64	23/64
$f(p)$	40610	1	141	1	1361	1	74	1
p	24/64	25/64	26/64	27/64	28/64	29/64	30/64	31/64
$f(p)$	14253	1	75	1	1589	1	148	1
p	32/64	33/64	34/64	35/64	36/64	37/64	38/64	39/64
$f(p)$	94792	1	30	1	1171	1	33	1
p	40/64	41/64	42/64	43/64	44/64	45/64	46/64	47/64
$f(p)$	4861	1	104	1	576	1	37	1
p	48/64	49/64	50/64	51/64	52/64	53/64	54/64	55/64
$f(p)$	35324	1	35	1	614	1	38	1
p	56/64	57/64	58/64	59/64	60/64	61/64	62/64	63/64
$f(p)$	6819	1	39	1	734	1	42	1

Table: $n = 7$, p for probability and $f(p)$ for number of preferences of probability p.

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See any patterns?

Introducing Middle School Math

Example. $n = 6$, the 16 preferences have probability $\frac{k}{32}$, where k is odd are:
Here we write $(a_1, a_2, a_3, a_4, a_5, a_6)$ as $a_1a_2a_3a_4a_5a_6$.

222222,
332222, 333222, 333322, 333332,
443222, 443322, 443332, 444322, 444332, 444432,
554322, 554332, 554432, 555432,
665432.

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Pascal's Triangle!

We Are Going Somewhere

Theorem 1.4. *There is one and only one parking preference for which the probability that every car parks is* $\frac{2t-1}{2^{n-1}}$*, where* $t \in [1, 2^{n-2}]$.

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About proving Theorem 1.4...

Flashback to the Olympiads

- **Lemma 4.2.1.** *One of the parking process of the preference satisfies: for every c_i, a_i is taken and $a_i - 1$ is open.*
- **Lemma 4.2.2.** *Denote b_i as the spot c_i actually parks, then $|b_i - b_j| = 1$ for some $j < i$, $i \in \{2, 3, 4, ..., n\}$.*
- **Lemma 4.2.3.** *The preference $(a_1, a_2, ..., a_n)$ satisfies: $0 \leq a_i - a_{i+1} \leq 1$.*
- **Lemma 4.2.4.** $a_n = 2$.
- **Lemma 4.2.5.** *There are 2^{n-2} preferences satisfying the lemmas above.*
- **Lemma 4.2.6.** *For each preference, c_i where $a_i \neq 2$ must take the only correct choice in order to keep c_j where $j > i$ still have choices to make, in order to achieve probability $\frac{k}{2^{n-1}}$ where k is odd for that parking process.*
- **Lemma 4.2.7.** *Preference of length n that consists of only 2s have probability $\frac{2^{n-1}-1}{2^{n-1}}$.*
- **Lemma 4.2.8** *The 2^{n-2} preferences described above all have probability $\frac{k}{2^{n-1}}$ where k is odd.*
- **Lemma 4.2.9.** *These 2^{n-2} probabilities are all different.*

Flashback to the Olympiads

To prove Lemma 4.2.9. from the last slide.

Theorem 1.5. *For parking preference* $(a_{k_1}, \ldots, a_{k_2}, \ldots, a_{k_3}, \ldots, a_{k_{a_{k_1}}-1}, \ldots, a_n)$, *where* $a_j = a_{j+1}$ *when* $j \in [k_i, k_{i+1} - 2]$ *and from* $a_{k_{a_{k_1}}-1}$ *to* a_n *they are all 2s, (The only form qualifies a probability* $\frac{2t-1}{2^{n-1}}$, *where* $t \in [1, 2^{n-2}]$*) has probability* $\frac{\ell}{2^{n-1}}$.

$$\ell = 2^{n-1} - 2^{k_{a_{k_1}-1}-1} + \sum_{j=2}^{a_{k_1}-1} 2^{k_j-2} - 2^{k_{j-1}-1}$$

Example 4.3.1. For parking preference (8,8,7,6,5,5,5,4,3,2,2,2), $\ell = 2^{11} - 2^9 + 2^6 - 2^4 + 2^1 - 2^0$, and the probability that every car parks is $\frac{\ell}{2^{11}} = \frac{1585}{2048}$.

Thank you!