A Combinatorial Proof for the Aftermath of a Party

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Definition

aftermath
/'æftəɹ,mæθ/

- the consequences of an event, especially a catastrophic event.
- (analogous to afterparty) the math that happens after a party.

Party Scene

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- "This card is for University identification and must be carried by the named individual at all times." **(existence)**

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- Given that there is no fake ID **(uniqueness)**
- What is the expected number of people who grab their own ID?

In other words

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- This talk:
We got a longer proof!

Like, just, why?

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- **Read like a writer**

To actually count this

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- That # is
  $\binom{n}{k}(\#\text{ s.t. for the rest } n-k \text{ people no one gets their own ID})$

And that #... (for no one gets their own ID)

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- $f(n) = (f(n-1) + f(n-2))(n-1)$, here we go

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$$\sum_{k=1}^{n} k \binom{n}{k} f(n-k)$$

to be $n!$

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- we want
$$\sum_{k=1}^{n} k \binom{n}{k} f(n-k)$$
to be $n!$
- We know this
$$\sum_{k=0}^{n} \binom{n}{k} f(n-k)$$
is $n!$

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- Observe:

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- The details are messing with combinatorial identities that no one wants to see.

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- **Hey! 21 and 15**
- (we got to split into stuff from Pascal's triangle)

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