Erdős–Ko–Rado Problems

Melanie Tian

Lake Forest College

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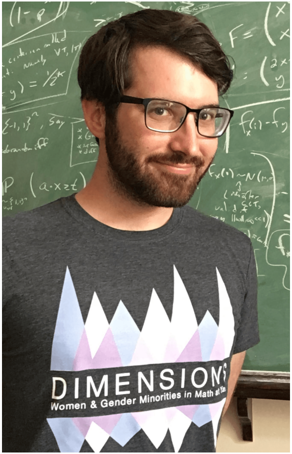
REU Polymath Jr.

Professor Pat Devlin, Yale University



Combinatorics

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- “Counting”

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What is combinatorics?

- "Counting"
- "Anything that is not algebra, geometry, or number theory"
- "The full scope of combinatorics is not universally agreed upon" (Wikipedia)

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- Addictive Combinatorics

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- ...
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Extremal Combinatorics

How big (or small) can a finite arrangement be if we insist that we need to satisfy some restrictions?

Erdős–Ko–Rado

Problems considering intersecting set families.

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- **Erdős–Ko–Rado Theorem**
- F is a family of distinct subsets of $\{1, 2, \ldots, n\}$ such that each subset is of size k, with $n \geq 2k$,
- each pair of subsets has a nonempty intersection,
- then $|F| \leq \binom{n-1}{k-1}$.

Intro to Our Problem

A **subsequence** of a permutation:
anything you get by deleting stuff.

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- 2^n, duh.

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- say $\sigma = 12345$, $\tau = 13425$,
- then $LCS(\sigma, \tau) = 4$,
- because they both have 1345, and they don't have any common subsequences of length 5 ($\sigma \neq \tau$).

Our Problem

Let $f_k(n)$ denote the size of the biggest family of permutations of $\{1, 2, \ldots, n\}$ such that any two elements in our family have LCS at least k, find $f_k(n)$.

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- $f_{k+1}(n) \leq f_k(n)$

  (This is like how to get 90% on MATH-230 final? Leave an answer blank.)

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- How?
- Insist that everyone needs to have 123 as a subsequence.
- Now "build around" 123, you have 4 choices for where to put 4, and 5 choices for where to put 5, etc.

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- For example, let $n = 10$, $k = 3$. Observe that from every permutation of $\{1, 2, \ldots, 9\}$, we get 10 new ones.
- How?
- Because it literally doesn't matter where you put 10. (We've already satisfied all the requirements, TOTAL FREEDOM)

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- One standard move is you consider "pairs".
- Pair everyone with its reverse, e.g. 13245 with 54231. They don't share anything.

What if we don't solve the problem?

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Conjecture. $f_k(n) \leq \frac{n!}{k!}$

- Which is saying $f_k(n) = \frac{n!}{k!}$
- Maybe you can prove it.

Real-World Applications

Please, don't ask.